Centers and Centralizers

Definition

If G is a group, the subset of all elements g in G that commute with every other element of G (with respect to the operation of G) is called the **center** of the group, denoted $Z(G)$. That is,

$$Z(G) = \{g \in G \ : \ gx = xg \ \text{ for all } x \in G\}$$

Theorem 1

The center of a group G is a subgroup of G.

Note that the center of a group is never empty - the identity element of any group commutes with every other element of that group, so the center must at least contain the identity.

If a group G is Abelian, then the center of the group is the entire group: $Z(G) = G$.

Definition

If g is an element of a group G, the **centralizer** of g, denoted $C(g)$, is the set of all elements in G that commute with g (under the group operation). That is,

$$C(g) = \{x \in G \ : \ gx = xg\}$$

Theorem 2

The centralizer of an element g in a group G is a subgroup of G.

Since the identity e of a group always commutes with every other element, then the centralizer of e is equal to the entire group: $C(e) = G$.

If a group G is Abelian, then the centralizer of every group element g is the entire group: $C(g) = G$.

Theorem 3

The center of a group G is the intersection of the centralizer of every element in the group:

$$Z(G) = \bigcap_{g \in G} C(g)$$