Partial Fractions

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Many problems in mathematics can be solved with the help of the **method of partial fractions**. It is a method for decomposing a rational function into the sum of "simpler" rational functions. Recall, that a rational function is simply the quotient of two polynomials. Specifically, a function f is rational if it has the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

Method (*for the decomposition of* $\frac{p(x)}{q(x)}$ *into partial fractions*)

1. **Divide if improper.** If the degree of the numerator, $p(x)$, is greater than or equal to the degree of the denominator, $q(x)$, divide (by long division) $p(x)$ by $q(x)$ to obtain:

   $$\frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$$

   where s and r are polynomials and the degree of the remainder, $r(x)$, is less than the degree of $q(x)$. Go on and apply the next steps to find the decomposition of $\frac{r(x)}{q(x)}$.

2. **Factor the denominator.** Completely factor the denominator $q(x)$ into factors of the form

   $$(ax+b)^n \quad \text{and} \quad (ax^2+bx+c)^n$$

   where the quadratic ax^2+bx+c is irreducible. The denominator then is factored into the product of linear factors and quadratic factors, some of which are possibly repeated.

3. **Write the decomposition accordingly.** Write an equation of the form

   $$\frac{p(x)}{q(x)} = (\textit{the decomposition})$$

   I. **Case I - The denominator $q(x)$ is a product of distinct linear factors.**

   $$q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$$

   where no factor is repeated (and no factor is a constant multiple of another). Then there exist constants $A_1, A_2, \ldots, A_k$ such that:

   $$\frac{p(x)}{q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

   The A_i's are called the coefficients and are to be determined.

   II. **Case II - $q(x)$ is a product of linear factors, some of which are repeated.**
   Suppose the factor (a_1x+b_1) is repeated n times; that is $(a_1x+b_1)^n$ occurs in the factorization of $q(x)$. Then instead of the single term $A_1/(a_1x+b_1)$ above, we use:

   $$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_n}{(a_1x+b_1)^n}$$

   III. **Case III - $q(x)$ contains irreducible quadratic factors, none of which are repeated.**
   For each factor (ax^2+bx+c), the decomposition will include a term of the form

   $$\frac{Ax+B}{(ax^2+bx+c)}$$

   The A's and B's are called the coefficients and are to be determined.

   IV. **Case IV - $q(x)$ contains a repeated irreducible quadratic factor.**
   For each factor of the form $(ax^2+bx+c)^n$, include in the decomposition the following sum

   $$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

4. **Solve for the coefficients.** Using algebra, the substitution of the roots of the linear factors and other convenient values for x, and equating the coefficients of like powers, determine the A's and B's of the decomposition.

PROPER/IMPROPER

The function f is *proper* if the degree of the numerator p is strictly less than the degree of the denominator q, $\deg(p) < \deg(q)$.
On the other hand, the function f is *improper* if $\deg(p) \geq \deg(q)$.

GENERAL REMARKS

In an expression like:

$$\frac{x+5}{x^2+x-2} \quad = \quad \frac{2}{x-1} - \frac{1}{x+2}$$

The two terms on the right are called the *partial fractions*, for which the method is named.

We are still using the rational function f,

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

FUN FACTS

1. Any improper rational function can be written as the sum of a polynomial and a **proper** rational function.

   $$f(x) \;=\; \frac{p(x)}{q(x)} \;=\; s(x) + \frac{r(x)}{q(x)}$$

   where s and r are also polynomials.

2. Any **proper** rational function can be written as the sum of proper partial fractions.

3. All polynomials can be factored as a product of linear and irreducible quadratic factors[1].

[1] a result of the Fundamental Theorem of Algebra