INTEGRATION STRATEGY

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As we have seen, integration is more challenging than differentiation. In finding the derivative of a function it is obvious which differentiation formula we should apply. But it may not be obvious which technique we should use to integrate a given function, and in general, most integrals can be worked in more than one way.
Until now individual techniques have been applied in each section. No hard and fast rules can be given as to which method applies in a given situation, but I might suggest the following integration strategy. Note, this is not a set of rules or an algorithm, simply some guidelines to help you formulate your own integration strategy.

0. **Is it *obvious*?** (one of these:)

Table of basic Integration Formulas

$\int \mathrm{d}u = u + C$	$\int u^r \,\mathrm{d}u = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$
$\int e^u \,\mathrm{d}u = e^u + C$	$\int \frac{1}{u} \,\mathrm{d}u = \ln\lvert u\rvert + C$
$\int \sin u \,\mathrm{d}u = -\cos u + C$	$\int \cos u \,\mathrm{d}u = \sin u + C$
$\int \sec^2 u \,\mathrm{d}u = \tan u + C$	$\int \sec u \tan u \,\mathrm{d}u = \sec u + C$
$\int \frac{1}{a^2+u^2} \,\mathrm{d}u = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$	$\int \frac{1}{\sqrt{a^2-u^2}} \,\mathrm{d}u = \sin^{-1}\frac{u}{a} + C$

1. **Algebraically rearrange?** - Simplify the integrand if possible - Often the use of algebraic manipulation or trigonometric identities will simplify the integrand and make the method of integration obvious. (case 0), like:

   a) $\int \sqrt{x}(1+\sqrt{x}) \,\mathrm{d}x \quad = \quad \int \left(x^{1/2} + x\right) \,\mathrm{d}x$ or

   b) $\int (\sin x + \cos x)^2 \,\mathrm{d}x \quad = \quad \int (1 + 2\sin x \cos x) \,\mathrm{d}x$

2. **Will u substitution work?** – try before anything else below (Keep in mind, u-substitution undoes the chain rule.).

3. **Classify the integrand according to its form** (after considering steps 0,1,2)

   (a) Trig functions - product of powers of $\sin x$ & $\cos x$ or $\sec x$ & $\tan x$ - then use methods of section 7.3,

   (b) Rational functions - think **partial fraction decomposition**,

   (c) Integration by parts if the integrand is a product of a power of x (or a polynomial) and a transcendental function (trig, exponential, $\ln$) then really think IBP's (Keep in mind, I.P.P's undoes the product rule.):

$$\int u \,\mathrm{d}v \;=\; uv - \int v \,\mathrm{d}u\,,$$

   Rule of Thumb: choose the 'u' in this order (LIATE): **L**ogarithm, **I**nverse trig, **A**lgebraic, **T**rigonometric, **E**xponential.

   (d) Radicals of the form: $\sqrt{\pm x^2 \pm a^2}$ or expressions in the form: $\left(\pm x^2 \pm a^2\right)^{p/q}$ **trig substitution**.

4. **Try again**, We only have two techniques of integration: substitution and integration by parts.
   Did you actually *try substitution*? Sometimes even if no substitution is obvious, some ingenuity (or desperation) may suggest an appropriate substitution **WRITE DOWN STUFF!**.
   Manipulate the integrand, in particular, reconsider the rearrangement of the integrand, including our favorites:
   multiply by one in a funny form and *adding zero in a funny way*.

$$\int \frac{1}{1-\cos x} \,\mathrm{d}x \;=\; \int \frac{1}{1-\cos x}\left(\frac{1+\cos x}{1+\cos x}\right) \mathrm{d}x \quad=\quad \int \frac{1+\cos x}{1-\cos^2 x} \,\mathrm{d}x \quad=\quad \int \left(\frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\right) \mathrm{d}x$$

▶ RATIONALIZING SUBSTITUTIONS

Some nonrational functions can be changed into rational functions using a appropriate substitution. For example, evaluate:

$$\int \frac{\sqrt{x+4}}{x}\,\mathrm{d}x$$

Make the substitution

$$u = \sqrt{x+4}\,;\ \text{ then }\ u^2 = x+4\,,\text{ so }\ \ x = u^2-4 \text{ and }\ \mathrm{d}x = 2u\,\mathrm{d}u.$$

thus

$$\int \frac{\sqrt{x+4}}{x}\,\mathrm{d}x \ = \ \int \frac{u}{u^2-4}\,2u\,\mathrm{d}u \ = \ 2\int\left(1+\frac{4}{u^2-4}\right)\,\mathrm{d}u$$

▶ COMPLETING THE SQUARE

Another idea that is helpful in solving certain problems is called **completing the square**. It involves rewriting a quadratic ax^2+bx+c as the difference of two squares by adding and subtracting $(b/2)^2$.

$$ax^2+bx+c = x^2+bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c = \left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c$$

The "two squares" are

$$x+\frac{b}{2} \quad\text{and}\quad \sqrt{\left(\frac{b}{2}\right)^2+c}.$$

▶ COMBINING ALGEBRAIC TECHNIQUES

For example:

$$\int \frac{x^3+4x^2-21}{x^2+6x+10}\,\mathrm{d}x \tag{1}$$

Substitution **nor** I.B.P.'s helps, and after polynomial long division we get:

$$\int \frac{x^3+4x^2-21}{x^2+6x+10}\,\mathrm{d}x \quad = \quad \int\left(x-2+\frac{2x-1}{x^2+6x+10}\right)\,\mathrm{d}x \tag{2}$$

Again, substitution does not immediately seem useful, so look at P.F.D.; but $x^2+6x+10$ is an irreducible quadratic, so there is no decomposition. On our integration strategy, we just hit number 4: try again.

Back to u-subs, if we let $u = x^2+6x+10$ then $du = (2x+6)dx$; we can get this in the numerator (add zero: $2x-1\ +6-6$), thus we can rewrite the RHS of (2) as:

$$\int (x-2)\,\mathrm{d}x + \int \frac{2x+6}{x^2+6x+10}\,\mathrm{d}x + \int \frac{-7}{x^2+6x+10}\,\mathrm{d}x \tag{3}$$

The two on the left we can now integrate. To put the third integral in integration friendly form, complete the square on the denominator $\{\ x^2+6x+10 = (x+3)^2+1\ \}$, so that (3) becomes:

$$\int (x-2)\,\mathrm{d}x + \int \frac{2x+6}{x^2+6x+10}\,\mathrm{d}x + \int \frac{-7}{(x+3)^2+1}\,\mathrm{d}x \tag{4}$$

All of these integrals in (4) we can now integrate, so an antiderivative of (1):

$$\int \frac{x^3+4x^2-21}{x^2+6x+10}\,\mathrm{d}x \quad = \quad \frac{x^2}{2}-2x+\ln\left(x^2+6x+10\right)-7\tan^{-1}(x+3)+C \tag{5}$$