Math 171 - 2007 Integration Review I

Evaluate the following integrals. The point here is not to have someone tell you which method to use, but to learn to decide for yourself. It is natural to try one technique and have it fail. Struggle with the ones you find difficult for a while before you seek help.

1. $\int \frac{1+x-x^2}{x^2}\,dx = \int (x^{-2}+x^{-1}-1)\,dx = -x^{-1}+\ln|x|-x+C = \frac{-1}{x}+\ln|x|-x+C$

2. $\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}(x^2)+C$
   $u=x^2$, $du=2x\,dx$

3. $\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln(x^2+1)+C$
   $u=1+x^2$, $du=2x\,dx$

4. $\int \frac{1+x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx = \tan^{-1}x + \frac{1}{2}\ln(x^2+1)+C$

5. $\int \tan^2\theta\sec^2\theta\,d\theta = \int u^2\,du = \frac{u^3}{3}+C = \frac{1}{3}\tan^3\theta + C$ ($u=\tan\theta$ $du=\sec^2\theta\,d\theta$)

6. $\int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx = \int \sec u\tan u\,du = \sec u + C = \sec(\ln x)+C$
   $u=\ln x$ $du=\frac{1}{x}dx$

7. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u+C = 2e^{\sqrt{x}}+C$
   $u=\sqrt{x}$ $du=\frac{1}{2\sqrt{x}}dx$

8. $\int e^t\sin e^t\,dt = \int \sin u\,du = -\cos u + C = -\cos(e^t)+C$
   $u=e^t$ $du=e^t dt$

9. $\int \sin\theta\cos\theta\,d\theta = \int u\,du = \frac{u^2}{2}+C = \frac{1}{2}\sin^2\theta + C$
   $u=\sin\theta$, $du=\cos\theta\,d\theta$

10. $\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x}{1+(e^x)^2}\,dx = \int \frac{1}{1+u^2}\,du = \tan^{-1}u + C = \tan^{-1}(e^x)+C$
    $u=e^x$ $du=e^x dx$

11. $\int \frac{\sec\theta\tan\theta}{1+\sec\theta}\,d\theta = \int \frac{1}{u}\,du = \ln|u|+C = \ln|1+\sec\theta|+C$
    $u=1+\sec\theta$

12. $\int \frac{4e^{2x}-4e^{4x}}{\sqrt{1-e^{4x}}}\,dx = \int \frac{4e^{2x}}{\sqrt{1-e^{4x}}}\,dx + \int \frac{-4e^{4x}}{\sqrt{1-e^{4x}}}\,dx = 2\int \frac{1}{\sqrt{1-u^2}}\,du + \int w^{-1/2}\,dw = 2\sin^{-1}u + 2w^{1/2}+C$
    $= 2\sin^{-1}(e^{2x}) + 2\sqrt{1-e^{4x}}+C$
    $u=e^{2x}$, $du=2e^{2x}dx$ $w=1-e^{4x}$, $dw=-4e^{4x}dx$

13. $\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx = \int e^u\,du = e^u + C = e^{\tan^{-1}x}+C$
    $u=\tan^{-1}x$

14. $\int_1^2 \frac{e^{1/x}}{x^2}\,dx = -\int_1^{1/2} e^u\,du = e^u\Big|_{1/2}^{1} = e - e^{1/2}$
    $u=\frac{1}{x}$

15. $\int_1^4 \frac{1}{\sqrt{y}(\sqrt{y}+1)^3}\,dy = 2\int_2^3 \frac{1}{u^3}\,du = 2(\tfrac{-1}{2})u^{-2}\Big|_2^3 = -\frac{1}{u^2}\Big|_2^3 = \left(\frac{1}{4}-\frac{1}{9}\right) = \frac{5}{36}$
    $u=\sqrt{y}+1$

16. $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx = \int_1^4 \frac{1}{\sqrt{u}}\,du = 2u^{1/2}\Big|_1^4 = 2(\sqrt{4}-\sqrt{1}) = 2$
    $u=\ln x$

17. $\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} u\,du = \frac{u^2}{2}\Big|_0^{\pi/6} = \frac{1}{2}\left(\frac{\pi}{6}\right)^2 - 0 = \frac{\pi^2}{72}$
    $u=\sin^{-1}x$

18. $\int_{-3}^0 (8-2x)\sqrt{9-x^2}\,dx = \int_{-3}^0 8\sqrt{9-x^2}\,dx + \int_{-3}^0 -2x\sqrt{9-x^2}\,dx$ $u=9-x^2$ $du=-2x\,dx$
    $u(0)=9$
    $= 8\frac{1}{4}\pi r^2 + \int_0^9 \sqrt{u}\,du$ $u(-3)=0$
    $= 2\pi(3)^2 + \frac{2}{3}u^{3/2}\Big|_0^9 = 18\pi + \frac{2}{3}(9^{3/2}-0) = 18\pi + \frac{2}{3}(27) = 18\pi + 18$